Solution 12

Supplementary Problems

1. (Optional) Let Ω be a region in space which is bounded by a smooth closed surface S.

   (a) Use the divergence theorem to derive the formula of volume of Ω:

   $$|\Omega| = \frac{1}{3}\iint_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n}\, d\sigma\ ,$$

   where $\mathbf{n}$ is the outer unit normal at S.

   (b) Assume that Ω is contained in a ball of radius R. Derive the inequality

   $$|\Omega| \le \frac{1}{3}R|S|\ ,$$

   where $|S|$ is the surface area of S.

   (c) Find a region Ω so that the inequality in (b) becomes equality.

   Solution.

   (a) We choose the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $\nabla \cdot \mathbf{F} = 3$ and, by the divergence theorem,

   $$\iint_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n}\, d\sigma = \iiint_\Omega \nabla \cdot \mathbf{F}\, d\sigma = 3|\Omega|\ .$$

   (b) Using the inequality $|\mathbf{a} \cdot \mathbf{b}| \le |\mathbf{a}||\mathbf{b}|$ for vectors $\mathbf{a}, \mathbf{b}$, we have

   $$|(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n}| \le |\mathbf{x}||\mathbf{n}| \le R\ .$$

   It follows that

   $$\begin{aligned} |\Omega| &\le \frac{1}{3}\left|\iint_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n}\, d\sigma\right| \\ &\le \frac{1}{3}R\left|\iint_S d\sigma\right| \\ &= \frac{1}{3}R|S|\ . \end{aligned}$$

   (c) Take S to the sphere of radius R with center at the origin. Then the right hand side of this inequality becomes $4\pi R^3/3$ which is equal to the left hand side of this inequality.